RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College affiliated to University of Calcutta)

M.A./M.Sc. FOURTH SEMESTER EXAMINATION, MAY 2023

SECOND YEAR (BATCH 2021-23)

Date : 17/05/2023 **MATHEMATICS**

Time : 11.00 am – 1.00 pm **Paper : CC18** Full Marks : 50

Group : A

Answer **any one** question: [1×7]

1. Reduce the following equation to its canonical form:

$$y^2u_{xx} - 2xy\,u_{xy} + x^2u_{yy} = \frac{y^2}{x}u_x + \frac{x^2}{y}u_y$$

2. Reduce the equation $u_{xx} - 4x^2u_{yy} = \frac{1}{x}u_x$ to its canonical form and hence solve it.

Answer **any three** questions: [3×6]

3. If the linear second order partial differential equation is of the form $Lu = 0$ where $L \equiv \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c,\ a_{ij} = a_{ji}, b_i$ and c are functions of $x_1, x_2, ..., x_n$ having continuous second order partial derivatives in $D \subset E^n$ then find the condition for which L will be self-adjoint operator.

4. Using the method of characteristics find the D'Alembert's solution of $\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$ where $u(x,0) = f(x), \frac{\partial u}{\partial t}(x,0) = g(x)$. Also prove that the solution is unique.

5. Show that the solution of the equation $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0;\ r \ge a,\ 0 \le \theta \le 2\pi$ is $u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(r^2 - a^2) f(\varphi)\,d\varphi}{r^2 - 2ar\cos(\varphi - \theta) + a^2}$ where $u(a,\theta) = f(\theta), 0 \le \theta \le 2\pi$ and u is bounded as $r \to \infty$.

6. Solve : $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t},\ 0 \le x \le a,\ 0 \le y \le b,\ t \ge 0;$ with boundary conditions $T(0,y,t) = T(a,y,t) = 0, 0 \le y \le b,\ t \ge 0;\ T(x,0,t) = T(x,b,t) = 0, 0 \le x \le a, t \ge 0$ and the initial condition $T(x,y,0) = f(x,y), 0 \le x \le a, 0 \le y \le b.$

7. By the method of separation of variables, solve the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

Group : B

Answer **any five** questions :- [5×5]

8. If $P_n(x)$ is a Legendre polynomial of degree n, establish the recurrence relation : $P_n'(x) - 2xP_{n-1}'(x) + P'_{n-2}(x) - P_{n-1}(x) = 0, n = 1, 2, 3........$

9. Prove that $\int_{-1}^{1}(1-x^2)P_l'(x)P_m'(x)\,dx = \dfrac{2l(l+1)}{2l+1}\delta_{lm}$.

10. Evaluate : $\int_0^x x^3 J_0(x)\,dx,$ where $J_0(x)$ is the Bessel's function of order 0.

11. Show that the integral representation of $J_n(x)$ is given by $J_n(x) = \dfrac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta - n\theta)\,d\theta.$

12. Prove that (i) $xF(1,1;2;-x) = \ln(1+x)$ and

    (ii) $xF\left(\dfrac{1}{2},1;\dfrac{3}{2};-x^2\right) = \tan^{-1}x,$ where F is a hypergeometric function. [3+2]

13. Express $H(x) = x^4 + 2x^3 + 2x^2 - x - 3$ in terms of Hermite polynomials.

14. Show that $\dfrac{1}{1-t}e^{-\frac{xt}{1-t}} = \sum_{n=0}^{\infty} L_n(x)t^n,$ where $L_n(x)$ is a Laguerre polynomial of order n.

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